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Magnetic susceptibility of relativistic Fermi gas

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Abstract. The magnetic susceptibility of a degenerate relativisitc Fermi gas is obtained for charged fermions with anomalous magnetic moments.

The expression for the magnetic susceptibility of the non-relativistic degenerate electron gas is well known in the theory of metals (see for example Ziman (1972)). It consists of two parts corresponding to two different physical effects. The first of them is connected with the behaviour of spin moments in the external magnetic field (Pauli paramagnetism (Pauli 1927)). The second is the result of the circular motion of charges in the magnetic field (Landau diamagnetism (Landau 1930)). For non-relativistic fermions the paramagnetic and diamagnetic susceptibilities can be independently calculated. This is not true for the relativistic case. Magnetic properties of the relativistic Fermi gas are of interest because degenerate relativistic electrons and nucleons form white dwarfs and neutron stars (see for example Weinberg (1972)). As is known, the nucleons have anomalous magnetic moments which also contribute to the magnetic susceptibility. In this connection we consider the Lagrangian (Gell-Mann 1956, see also Bjorken and Drell 1965) including the dipole interaction between the fermion ψ and the external electromagnetic field A_μ,

$$L = \psi\{\mathrm{i}\gamma_\mu(\partial_\mu - \mathrm{i}qeA_\mu) + \mathrm{i}(ke/8M)[\gamma_\mu, \gamma_\nu]F_{\mu\nu} - M\}\bar{\psi}, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, M is the fermion mass, $-e$ is the charge of an electron and γ_μ are Dirac matrices. For an electron $q = -1$, $k = 0$. For a proton $q = +1$, $k = +1.79$. For a neutron $q = 0$, $k = -1.91$.†

In a constant homogeneous magnetic field

$$\boldsymbol{H} = \mathrm{curl}\,\boldsymbol{A} = \mathrm{constant}, \qquad A_0 = 0, \tag{2}$$

the energy levels for the Lagrangian (1) were obtained exactly (Tsai and Yildiz 1971):

$$\varepsilon^2 = p_H^2 + \{(ke/2M)\sigma H + [M^2 + |q|eH(2l+1) + qe\sigma H]^{1/2}\}^2 \tag{3}$$

where p_H is the momentum projection on the magnetic field $\boldsymbol{H}$, $l = 0, 1, 2, \ldots$ and $\sigma = \pm 1$ are the orbital and spin quantum numbers.

The thermodynamic potential density of the fermion gas has the form

$$\Omega = -T \sum_i \ln\{1 + \exp[(\mu - \varepsilon_i)/T]\} \tag{4}$$

† The step of orbital quantisation in the magnetic field $2\pi|q|eH \to 0$ as $q \to 0$.

where ε_i are particle energy levels, μ is the chemical potential and T is temperature. In a weak magnetic field

$$\Omega = \Omega_0 - \tfrac{1}{2}\chi H^2 \tag{5}$$

where χ is the magnetic susceptibility and Ω_0 does not depend on the magnetic field. For the calculation of Ω, the sum (4) can be replaced by an integration over energy ε and summation over spin σ and orbital l quantum numbers

$$\sum_i \rightarrow \sum_{\sigma,l} \int \mathrm{d}\varepsilon\, \nu_\sigma(\varepsilon, l) \tag{6}$$

where ν_σ is the level density of fermions which have various spin projections in the direction of the field $\boldsymbol{H}$.

In the plane normal to the magnetic field, fermions move in closed orbits, their area being quantised in the momentum space according to the formula

$$S_l = \pi(2l+1)|q|eH. \tag{7}$$

Taking this into account, one can immediately obtain the level density

$$\nu_\sigma = [|q|eH/(2\pi)^2]\, \mathrm{d}p_H^{(\sigma)}/\mathrm{d}\varepsilon \tag{8}$$

where

$$p_H^{(\sigma)} = \pm(\varepsilon^2 - a_\sigma^2)^{1/2} \tag{9}$$

with

$$a_\sigma = \sigma(keH/2M) + \{M^2 + 2[|q|l + \tfrac{1}{2}(|q| + \sigma q)]eH\}^{1/2}. \tag{10}$$

Substituting (8) into (4), integrating by parts and considering T to be small compared with the Fermi energy ε_{F}, we have

$$\begin{aligned}\Omega &= -\frac{2|q|eH}{(2\pi)^2} \sum_{\sigma,l}^{l_{\max}^{(\sigma)}} \int_{a_\sigma}^{\varepsilon_{\mathrm{F}}} \mathrm{d}\varepsilon\, (\varepsilon^2 - a_\sigma^2)^{1/2} \\ &= -\frac{|q|eH}{(2\pi)^2} \sum_{\sigma,l}^{l_{\max}^{(\sigma)}} f_\sigma(l)\end{aligned} \tag{11}$$

where

$$f_\sigma(l) = \varepsilon_{\mathrm{F}}(\varepsilon_{\mathrm{F}}^2 - a_\sigma^2)^{1/2} - a_\sigma^2 \ln\{[\varepsilon_{\mathrm{F}} + (\varepsilon_{\mathrm{F}}^2 - a_\sigma^2)^{1/2}]/a_\sigma\} \tag{12}$$

and

$$l_{\max}^{(\sigma)} = [(\varepsilon_{\mathrm{F}} - \sigma keH/2M)^2 - M^2/2|q|eH] - (|q| + \sigma q)/2|q|. \tag{13}$$

In the case of a weak field H ($l_{\max}^{(\sigma)} \gg 1$), one may replace the sum in the expression (11) by an integral, using the Euler–McLaurin summation formula

$$\sum_{l=0}^{n} f(l) = \int_0^n f(l)\, \mathrm{d}l + \tfrac{1}{2}[f(0) + f(n)] - \sum_{m=1}^{\infty} \frac{(-1)^m B_m}{(2m)!}[f^{(2m-1)}(n) - f^{(2m-1)}(0)] \tag{14}$$

where $f^{(n)}(x) = \mathrm{d}^n f/\mathrm{d}x^n$ and B_m are the Bernoulli numbers. Using this formula, and after a number of simple transformations, we obtain to order H^2

$$\Omega = -\frac{|q|eH}{(2\pi)^2}\left(\sum_\sigma \int_{t_\sigma}^{l_{\max}^{(\sigma)}} f_\sigma(l)\, \mathrm{d}l + \sum_\sigma [(t_\sigma + \tfrac{1}{2}) f_\sigma(0) + \tfrac{1}{2}(t_\sigma^2 - \tfrac{1}{6}) f'_\sigma(0)]\right) \tag{15}$$

where we have used $f_\sigma(l_{\max}^{(\sigma)}) = f'_\sigma(l_{\max}^{(\sigma)}) = 0$ and

$$t_\sigma = -\frac{k+|q|+\sigma q}{2|q|} + \frac{k^2}{|q|}\frac{eH}{8M^2}. \tag{16}$$

It may be easily seen that the integral term in (15) can be written in the form

$$|q|eH \sum_\sigma \int_{t_\sigma}^{l_{\max}^{(\sigma)}} f_\sigma(l)\,\mathrm{d}l = \sum_\sigma \int_M^{e_F} f(a_\sigma)\left(a_\sigma - \sigma\frac{keH}{2M}\right)\mathrm{d}a_\sigma \tag{17}$$

and therefore it contribution to the thermodynamic potential Ω_0 is independent of the magnetic field.

The variation of the fermion thermodynamic potential when a weak field is switched on is given by the second sum in (15). Using the expressions (12) and (13) for $f_\sigma(l)$ and $l_{\max}^{(\sigma)}$, we find this variation of order H^2. The magnetic susceptibility is then given by

$$\chi = \frac{e^2}{4\pi^2}\left(\frac{k^2 v_F}{2(1-v_F^2)} + (\tfrac{1}{4}k^2 + kq + \tfrac{1}{3}q^2)\ln\frac{1+v_F}{1-v_F}\right) \tag{18}$$

where v_F is the Fermi velocity.

In the non-relativistic approximation ($v_F \ll 1$) the expression for susceptibility may be represented as a sum of paramagnetic and diamagnetic terms

$$\chi = (e^2 v_F/4\pi^2)[(k+q)^2 - \tfrac{1}{3}q^2]. \tag{19}$$

For $k=0$, $q=-1$ and these terms coincide with well known expressions (Pauli 1927, Landau 1930) for the electron gas.

The total variation of the thermodynamic potential connected with the appearance of the magnetic field inside the Fermi gas has the form

$$\delta\Omega = -\tfrac{1}{2}\chi H^2 + \tfrac{1}{2}H^2 \tag{20}$$

where the last term represents the self-energy of the magnetic field. The inequality $\delta\Omega > 0$ gives the well known electrodynamic condition

$$\chi < 1 \tag{21}$$

for the stability of matter with respect to the spontaneous magnetic field produced by the ordering of the spin and orbital moments.

The stability of the relativistic electron gas with respect to the spontaneous magnetisation was studied in Canuto and Chiu (1968). The present formula for susceptibility seems to indicate that their discussion is not correct. As can be seen from (18), the stability of the electron gas ($k=0$, $q=-1$) is lost in the ultrarelativistic (unrealistic as it is) limit $v_F \to 1$.

In connection with our results, we also note that the nucleon density phase transition into the state with $M \to 0$ ($v_F \to 1$) studied by Lee and Wick (1974) (see also Krive and Chudnovsky (1978)) can lead to the spontaneous magnetisation of the neutron star.

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